0112 – Free Vibrations of a String

Description

A thin string of diameter D is tensioned by the initial strain ε_0[1] and initially deflected at a distance a according to **Figure 1**. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.0	MPa
		Poisson's Ratio	ν	0.300	–
		Linear Density	μ	24.662	$\mathrm{kgm^{-1}}$
Geometry		Diameter	D	0.002	m
		Length	L	1.000	m
		Deflection Position	a	0.500	m
Load		Tension	ε_0	0.001	–
		Initial Deflection	h	0.020	m

Determine the deflection u_{zA} of the test point A ($x = a$) at given test times.

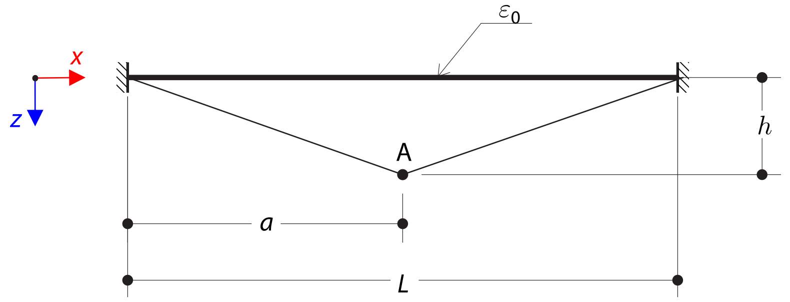

Figure 1: Problem Sketch

Analytical Solution

The general solution of string free vibrations was determined in Verification Example 0106. It can be written as a sum of the mode shapes

$$u_z(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{\Omega_n}{c}x\right)\left(A_n \cos(\Omega_n t) + B_n \sin(\Omega_n t)\right) \qquad (112-1)$$

[1] The initial strain ε_0 has to be assigned in DYNAM Pro Details as Minimum Axial Strain for Cables and Membranes ε_{min}.

where A_n and B_n are appropriate integration constants given by the initial conditions. In this example, the string is initialy deflected at distance a equalling $u_z(a,0) = h$. The function of initial deflection can then be written as

$$u_z(x,0) = u_{z0}(x) = \begin{cases} \dfrac{h}{a}x, & \text{for } x \in [0,a]; \\ \dfrac{h}{a-L}(x-L), & \text{for } x \in (a,L] \end{cases}$$

The initial velocity of the string is equal to zero

$$\frac{\partial u_z}{\partial t}(x,0) = v_{z0}(x) = 0, \quad \text{for } x \in [0,L] \tag{112 – 2}$$

Substituting $t = 0$ into **(112 – 1)** while considering the latter initial conditions, the values of A_n and B_n are obtained

$$u_{z0}(x) = u_z(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\Omega_n}{c}x\right) \tag{112 – 3}$$

$$v_{z0}(x) = \frac{\partial u_z}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \Omega_n \sin\left(\frac{\Omega_n}{c}x\right) \tag{112 – 4}$$

Multiplying both equations by the m-th mode function $X_m(x) = \sin\left(\frac{\Omega_m}{c}x\right)$ and integrating over the string length, the following relations are obtained[2]

$$\int_0^L u_{z0}(x)X_m(x)\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^L A_n X_n(x)X_m(x)\mathrm{d}x = A_m\frac{L}{2} \tag{112 – 5}$$

$$\int_0^L v_{z0}(x)X_m(x)\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^L \Omega_n B_n X_n(x)X_m(x)\mathrm{d}x = \Omega_m B_m\frac{L}{2} \tag{112 – 6}$$

From **(112 – 2)** and **(112 – 6)** there is $B_m = 0$. The constant A_m can be calculated according to **(112 – 5)**

$$A_m = \frac{2}{L}\int_0^L u_{z0}(x)\sin\left(\frac{\Omega_m}{c}x\right)\mathrm{d}x \tag{112 – 7}$$

Hence, the general solution of the string free vibrations is

$$u_z(x,t) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{\Omega_m}{c}x\right)\cos(\Omega_m t) \tag{112 – 8}$$

where the natural frequency of the string Ω_m is defined as

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[2] According to the ortogonality of the mode functions, it holds that $\int_0^L X_m(x)X_n(x)\mathrm{d}x = \frac{L}{2}\delta_{mn}$, where δ_{mn} is the Kronecker delta function.

$$\Omega_m = \frac{n\pi c}{L} \tag{112 – 9}$$

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.07.05 and RSTAB 8.07.01
- The string is modeled by the member type Cable
- The member is divided into 100 parts
- Isotropic linear elastic material model is used
- Shear stiffness of members is deactivated
- Subspace iteration method is used

Results

Structure Files	Program
0112.01	RFEM 5 – RF-DYNAM Pro
0112.02	RSTAB 8 – DYNAM Pro

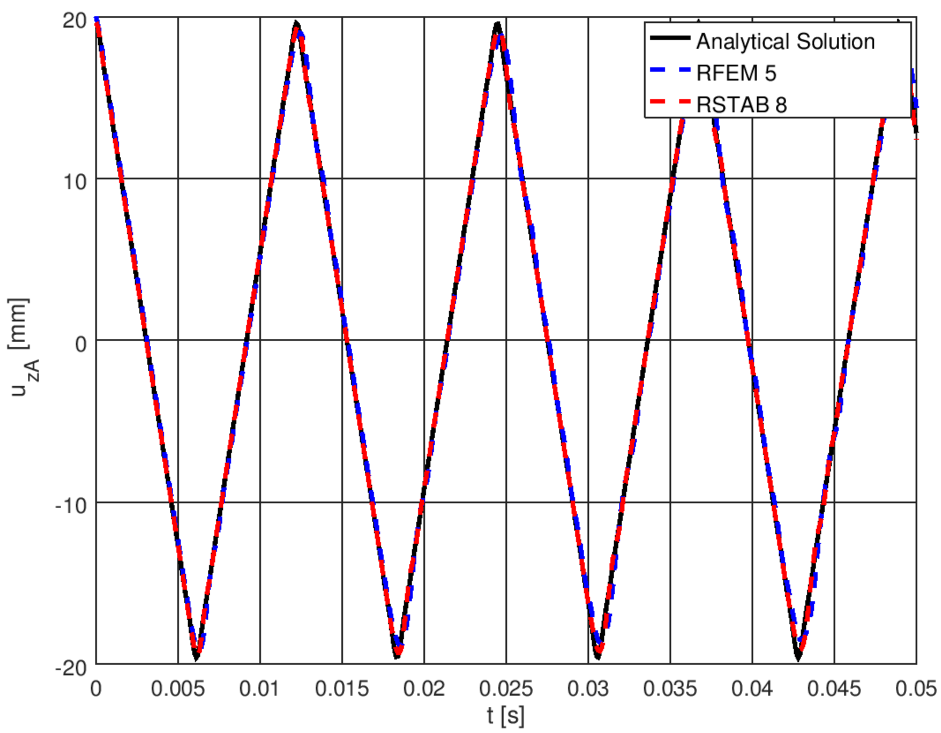

Figure 2: Comparison of analytical solution to the RFEM 5 / RSTAB 8 results

Test time	Analytical Solution	RFEM 5 RF-DYNAM Pro		RSTAB 8 DYNAM Pro	
	u_{zA} [mm]	u_{zA} [mm]	Ratio [-]	u_{zA} [mm]	Ratio [-]
$t_1 = 0.015$ s	1.862	2.165	1.163	1.854	0.996
$t_2 = 0.030$ s	-16.227	-15.761	0.971	-16.249	1.001
$t_3 = 0.040$ s	-1.683	-1.391	0.826	-1.731	1.029